Temporal evolution of product shock measures in the totally asymmetric simple exclusion process with sublattice-parallel update

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It is known that when the steady state of a one-dimensional multispecies system, which evolves via a random-sequential updating mechanism, is written in terms of a linear combination of Bernoulli shock measures with random-walk dynamics, it can be equivalently expressed as a matrix-product state. In this case the quadratic algebra of the system always has a two-dimensional matrix representation. Our investigations show that this equivalence exists at least for the systems with deterministic sublattice-parallel update. In this paper we consider the totally asymmetric simple exclusion process on a finite lattice with open boundaries and sublattice-parallel update as an example.

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I. INTRODUCTION

Recently, the steady-state properties of exclusion processes which belong to the class of driven-diffusive systems have been under much investigation because of their unique physical characteristics such as shocks and nonequilibrium phase transitions [1–3]. The matrix-product approach has turned out to be one of the most powerful techniques in determining the steady states of such stochastic nonequilibrium systems which have been used to model biological transport and traffic flow. The matrix-product approach has been interpreted from different points of views (a recent review of this approach can be found in [4]). According to this approach the nonequilibrium steady-state weight of any given configuration of a one-dimensional stochastic system can be considered as a matrix element of product of noncommuting operators, one for each lattice site, chosen according to the state of the lattice site. In order to calculate these weights one needs to know a set of algebraic relations between these operators. Whether these operators have matrix representations is a challenging issue.

We have recently investigated the relation between the dimensionality of the matrix representation of the algebra of a given one-dimensional driven-diffusive system with nearest-neighbor interactions and the possibility that the steady state of the system in question can be written in terms of a linear superposition of product shock measures. In this context a shock is defined as a sharp discontinuity in the density profile of particles on the lattice. In [5] we have shown that for the one-dimensional driven-diffusive systems defined on a discrete lattice of finite size in which a single product shock measure has a simple random-walk dynamics under the time evolution generated by the stochastic Hamiltonian of the system, the steady state of the system can easily be written as a linear combination of these single product shock measures. In most cases it is necessary that some constraints on the microscopic reaction rates of the system are satisfied. Surprisingly we have seen that at the same time the steady state of the system can be written as a matrix-product state with two-dimensional matrix representation for the algebra of the system, provided that the same constraints (if they exist) on the microscopic reaction rates are satisfied. The matrix representation in this case obeys an special structure which contains very detailed information about the hopping rates of the shock front, as well as the density of particles on the left- and the right-hand sides of the shock front. In the present work we aim to investigate the same issue; however, this time it is for the systems in a different updating scheme, i.e., the sublattice-parallel dynamics.

One of the simplest, yet interesting, driven-diffusive model which has been studied widely during recent years is the asymmetric simple exclusion process (ASEP). In this model the classical particles enter the system from the left boundary of a discrete lattice, diffuse in its bulk, and leave it from the right boundary with certain rates. The derivation of the matrix-product representation from the algebraic Bethe ansatz for this model has been studied in [6]. For this model the equivalence between the partition functions of the system with random-sequential dynamics and the partition function of a two-dimensional lattice path model of one-transit walks or Dyck paths has been studied in [7]. Under the parallel dynamics the partition function of the ASEP can be expressed as one of several equivalent two-dimensional lattice path models involving weighted Dyck paths [8]. The dynamics of a single shock front in the ASEP with a discrete time updating scheme defined on an infinite lattice has already been studied in [9].

In this paper we will answer the question whether the existence of a two-dimensional representation for the quadratic algebra of a driven-diffusive system with a discrete time updating (more specifically sublattice-parallel updating) and nearest-neighbor interactions is related to the fact that the steady state of the system can be constructed in terms of a linear superposition of product shock measures with simple random-walk dynamics. We will consider the totally asymmetric simple exclusion process known as TASEP with open boundaries as a simple example. In this model the particles only hop toward the right boundary after being injected into the lattice from the left boundary. As we will see, quite similar to the case of random-sequential updating scheme studied in [5], it seems that whenever the quadratic algebra of the

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system (since we only consider the systems with nearest-neighbors interactions) has a two-dimensional matrix representation with a specific structure (which will be discussed later), then we can conclude that the steady state of the system is made up of a linear combination of product shock measures with a shock front which has simple random-walk dynamics and vice versa. One of the differences here is that in the case of sublattice updating scheme one should define two different shocks which behave differently at even or odd lattice site. This has already been observed in [9] for an infinite system.

In the following sections we will first present the mathematical tools and definitions. Then we will study the time evolution of two product shock measures defined at even and odd sites under the sublattice-parallel update. Then we will construct the steady state of the system in terms of a linear combination of these shocks. We will bring the quadratic algebra of the system and its two-dimensional representation in terms of the shock characteristics. The conclusion will be presented in the last section.

II. MATHEMATICAL PRELIMINARIES

Let us start with the definitions and mathematical preliminaries. Consider a general two-state driven-diffusive system with nearest-neighbor interactions and sublattice-parallel dynamics in which classical particles move on a one-dimensional lattice of length $2L$ with open boundaries. The bulk dynamics consists of two half time steps. In the first half time step even lattice sites and also the first and the last lattice sites are updated. From the first and the last lattice sites, the particles can be injected or extracted with certain probabilities. In the second half time step only the odd lattice sites are updated. The corresponding transfer matrix T consists of two factors $T=T_2T_1$ defined as [10]

$$T_1=\mathcal{L}\otimes\mathcal{T}\otimes\cdots\otimes\mathcal{T}\otimes\mathcal{R}=\mathcal{L}\otimes\mathcal{T}^{\otimes(L-1)}\otimes\mathcal{R},$$

$$T_2=\mathcal{T}\otimes\mathcal{T}\otimes\cdots\otimes\mathcal{T}=\mathcal{T}^{\otimes L},$$

where $\mathcal{T}$, $\mathcal{L}$, and $\mathcal{R}$ are the matrices for bulk interactions, particle input, and particle output, respectively. The time evolution of the probability distribution is governed by the following equation:

$$T|P(t)\rangle=|P(t+1)\rangle. \tag{1}$$

In long-time limit the system approaches its steady state and its nonequilibrium probability distribution satisfies the following equation:

$$T|P^*\rangle=|P^*\rangle. \tag{2}$$

As a simple example consider the TASEP which in an appropriate basis the above transfer matrix can be written as [10]

$$\mathcal{T}=\begin{pmatrix}1&0&0&0\\0&1&1&0\\0&0&0&0\\0&0&0&1\end{pmatrix},\quad \mathcal{L}=\begin{pmatrix}1-\alpha&0\\\alpha&1\end{pmatrix},\quad \mathcal{R}=\begin{pmatrix}1&\beta\\0&1-\beta\end{pmatrix}. \tag{3}$$

As can be seen, the particles in the bulk of the lattice move only to the right deterministically while obeying the exclusion principle. The particles can enter (leave) the lattice only from the left (right) boundary with the probability $\alpha(\beta)$. The dynamics and also the steady state of the TASEP with open boundaries has been proposed and studied in [11]. Later its steady state was studied using a matrix formalism in [10]. In the thermodynamic limit, i.e., $L\gg1$ one finds that the system has two different phases: a high-density phase for $\alpha>\beta$ and a low-density phase for $\alpha<\beta$. A first-order phase transition also takes place at the transition line $\alpha=\beta$.

III. TEMPORAL EVOLUTION OF SHOCKS

In what follows we study the time evolution of two product shock measures using the time evolution equation (1). We consider a discrete lattice of length $2L$ and introduce two product shock measures at even sites $2k$ $(k=1,\ldots,L)$ as $|\mu_{2k}\rangle$ and at odd sites $2k+1$ $(k=0,\ldots,L)$ as $|\mu_{2k+1}\rangle$ according to

$$|\mu_{2k}\rangle=\underbrace{\begin{pmatrix}1-\rho_1^{odd}\\\rho_1^{odd}\end{pmatrix}}_{1}\otimes\begin{pmatrix}1-\rho_1^{even}\\\rho_1^{even}\end{pmatrix}\otimes\cdots\otimes\underbrace{\begin{pmatrix}1-\rho_1^{odd}\\\rho_1^{odd}\end{pmatrix}}_{2k-1}\otimes\underbrace{\begin{pmatrix}1-\rho_2^{even}\\\rho_2^{even}\end{pmatrix}}_{2k}\otimes\cdots\otimes\begin{pmatrix}1-\rho_2^{odd}\\\rho_2^{odd}\end{pmatrix}\otimes\underbrace{\begin{pmatrix}1-\rho_2^{even}\\\rho_2^{even}\end{pmatrix}}_{2L}, \tag{4}$$

$$|\mu_{2k+1}\rangle=\underbrace{\begin{pmatrix}1-\rho_1^{odd}\\\rho_1^{odd}\end{pmatrix}}_{1}\otimes\begin{pmatrix}1-\rho_1^{even}\\\rho_1^{even}\end{pmatrix}\otimes\cdots\otimes\underbrace{\begin{pmatrix}1-\rho_1^{even}\\\rho_1^{even}\end{pmatrix}}_{2k}\otimes\underbrace{\begin{pmatrix}1-\rho_2^{odd}\\\rho_2^{odd}\end{pmatrix}}_{2k+1}\otimes\cdots\otimes\begin{pmatrix}1-\rho_2^{odd}\\\rho_2^{odd}\end{pmatrix}\otimes\underbrace{\begin{pmatrix}1-\rho_2^{even}\\\rho_2^{even}\end{pmatrix}}_{2L}. \tag{5}$$

We should explain a couple of points here. First, one should note that the shock front for $|\mu_{2k}\rangle$ lies between the lattice sites $2k-1$ and $2k$ while the shock front for $|\mu_{2k+1}\rangle$ lies between the lattice sites $2k$ and $2k+1$. Second, the shock $|\mu_{2L+1}\rangle$ indicates a flat distribution of particles with densities ρ_1^{odd} and ρ_1^{even} at odd and even lattice sites, respectively. In this case the shock front can be considered to be between the lattice sites $2L$ and $2L+1$ from which the latter is considered as an auxiliary site. Let us consider $|\mu_{2k}\rangle$ and $|\mu_{2k+1}\rangle$ as two initial probability distributions and investigate their time evolutions using Eq. (1). Suppose that under some constraints on the reaction rates the dynamics of these two shocks are given by

$$T|\mu_{2k}\rangle = \delta_l|\mu_{2k-1}\rangle + \delta_r|\mu_{2k+1}\rangle + \delta_s|\mu_{2k}\rangle \quad \text{for} \quad 1 \le k \le L,$$

$$T|\mu_{2k+1}\rangle = \delta_l\delta_s|\mu_{2k}\rangle + \delta_r\delta_s|\mu_{2k+2}\rangle + \delta_r^2|\mu_{2k+3}\rangle + \delta_l^2|\mu_{2k-1}\rangle + (\delta_s + 2\delta_l\delta_r)|\mu_{2k+1}\rangle \quad \text{for} \quad 1 \le k \le L-1,$$

$$T|\mu_1\rangle = \delta_r\delta_s|\mu_2\rangle + \delta_r^2|\mu_3\rangle + (1-\delta_r\delta_s-\delta_r^2)|\mu_1\rangle,$$

$$T|\mu_{2L+1}\rangle = \delta_l\delta_s|\mu_{2L}\rangle + \delta_l^2|\mu_{2L-1}\rangle + (1-\delta_l\delta_s-\delta_l^2)|\mu_{2L+1}\rangle, \tag{6}$$

in which $\delta_s = 1-\delta_l-\delta_r$.

The first two equations in Eq. (6) have already been obtained in [9] for the ASEP on an infinite lattice; however, since the definition of the transfer matrix in this paper is based on [10], these two time evolution equations have exchanged places as compared to the corresponding equations (44) and (48) in [9]. For the TASEP with open boundaries these equations of motion are valid, provided that we have

$$\rho_1^{odd} = 0, \quad \rho_2^{odd} = 1-\beta,$$

$$\rho_1^{even} = \alpha, \quad \rho_2^{even} = 1,$$

$$\delta_r = \beta, \quad \delta_l = \alpha. \tag{7}$$

Note that Eqs. (6) give a closed set of time evolution equations for $|\mu_k\rangle$'s in which $k=1,\ldots,2L+1$. It is also interesting to note that in contrast to the ASEP, here there is no constraint on the microscopic reaction rates (the boundary rates α and β). Although the particles move deterministically toward the right boundary, the shock fronts hop both to the left and to the right. This is a direct result of the updating scheme.

IV. STEADY STATE

The simplicity of the time evolution equations (6) allows us to construct the steady state of the system $|P^*\rangle$. As can be seen, they are similar to the time evolution equations for a simple random walker moving on a finite lattice with reflecting boundaries; however, the random walker behaves differently when it lies at an even or an odd lattice site. By considering a linear superposition of the shocks as

$$|P^*\rangle = \frac{1}{Z}\sum_{k=1}^{2L+1} c_k|\mu_k\rangle \tag{8}$$

and requiring that Eq. (2) should be satisfied, one finds

$$c_{2k} = \delta_s\left(\frac{\delta_r}{\delta_l}\right)^{2k-1} \quad \text{for} \quad k=1,\ldots,L, \tag{9}$$

$$c_{2k+1} = \left(\frac{\delta_r}{\delta_l}\right)^{2k} \quad \text{for} \quad k=0,\ldots,L. \tag{10}$$

The normalization factor or the partition function of the system Z can be easily calculated as

$$Z = \sum_{k=1}^{2L+1} c_k = \frac{1}{\delta_l-\delta_r}\left(\delta_l(1-\delta_r) - \delta_r(1-\delta_l)\left(\frac{\delta_r}{\delta_l}\right)^{2L}\right). \tag{11}$$

Since the steady state of the system is unique, if one calculates the steady-state probability distribution of the system using the matrix-product approach, one should find the same distribution as Eq. (8).

V. MATRIX-PRODUCT APPROACH

Let us now investigate the steady-state probability distribution function of our general two-state model with sublattice-parallel dynamics and nearest-neighbor interactions using the matrix-product approach. According to this approach (and in this particular updating scheme) the steady-state probability distribution function of the system can be written as [10]

$$|P^*\rangle = \frac{1}{Z}\langle\langle W| \left[\begin{pmatrix}\hat{E}\\ \hat{D}\end{pmatrix} \otimes \begin{pmatrix}E\\ D\end{pmatrix}\right]^{\otimes L} |V\rangle\rangle, \tag{12}$$

in which the operators $\hat{E}$ and $\hat{D}$ (E and D) stand for the presence of a hole and a particle at odd (even) sites, respectively. The normalization factor Z is usually called the partition function and can easily be written in a grand canonical ensemble as

$$Z = \langle\langle W|\ (\hat{E}+\hat{D})^L(E+D)^L\ |V\rangle\rangle. \tag{13}$$

The operators $(\hat{E},\hat{D})$ and (E,D) besides the vectors $|V\rangle\rangle$ and $\langle\langle W|$ are acting in an auxiliary space. According to the standard matrix-product approach by requiring that Eq. (2) is satisfied one finds that the above-mentioned operators and vectors should satisfy a quadratic algebra given by [10]

$$\mathcal{T}\left[\begin{pmatrix}E\\ D\end{pmatrix} \otimes \begin{pmatrix}\hat{E}\\ \hat{D}\end{pmatrix}\right] = \begin{pmatrix}\hat{E}\\ \hat{D}\end{pmatrix} \otimes \begin{pmatrix}E\\ D\end{pmatrix},$$

$$\langle\langle W|\ \mathcal{L}\begin{pmatrix}\hat{E}\\ \hat{D}\end{pmatrix} = \langle\langle W| \begin{pmatrix}E\\ D\end{pmatrix},$$

$$\mathcal{R}\begin{pmatrix}E\\ D\end{pmatrix} |V\rangle\rangle = \begin{pmatrix}\hat{E}\\ \hat{D}\end{pmatrix} |V\rangle\rangle. \tag{14}$$

Surprisingly, one can see that the following two-dimensional matrix representation which can be written in terms of the shock hopping rates and the densities of the Bernoulli measures at the left- and the right-hand sides of the shock position can generate exactly the same probability distribution (8),

$$\hat{D}=\begin{pmatrix}\rho_2^{odd} & 0\\ \hat{d} & \frac{\delta_r}{\delta_l}\rho_1^{odd}\end{pmatrix},\quad \hat{E}=\begin{pmatrix}1-\rho_2^{odd} & 0\\ -\hat{d} & \frac{\delta_r}{\delta_l}(1-\rho_1^{odd})\end{pmatrix},$$

$$D=\begin{pmatrix}\rho_2^{even} & 0\\ d & \frac{\delta_r}{\delta_l}\rho_1^{even}\end{pmatrix},\quad E=\begin{pmatrix}1-\rho_2^{even} & 0\\ -d & \frac{\delta_r}{\delta_l}(1-\rho_1^{even})\end{pmatrix},$$

$$\langle\langle W| = (w_1,w_2),\qquad |V\rangle\rangle=\begin{pmatrix}v_1\\ v_2\end{pmatrix}, \tag{15}$$

provided that we have

$$v_1w_2d=\frac{(\delta_r-1)(\rho_2^{even}-\rho_1^{even})}{\frac{\delta_l}{\delta_r}-1},$$

$$v_1w_2\hat{d}=\frac{(\delta_l-1)(\rho_2^{odd}-\rho_1^{odd})}{\frac{\delta_l}{\delta_r}-1}. \tag{16}$$

These relations are nothing but two constraints on the parameters v_1, v_2, w_1, w_2, d, and $\hat{d}$; therefore, only four of these parameters are free. Note that the densities in the shock measures and also the shock front hopping rates in Eqs. (6) should be fixed by the boundaries and the microscopic reaction rates, therefore, are not free parameters.

Let us go back to our simple example. It is shown in [10] that the TASEP has a quadratic algebra which can be written as

$$[E,\hat{E}]=[D,\hat{D}]=0,\quad E\hat{D}=[\hat{E},D],$$

$$\hat{D}E=0,\quad \langle\langle W|\ \hat{E}(1-\alpha)=\langle\langle W|\ E,$$

$$\langle\langle W|\ (\alpha\hat{E}+\hat{D})=\langle\langle W|\ D,\quad (1-\beta)D\ |V\rangle\rangle=\hat{D}\ |V\rangle\rangle,$$

$$(E+\beta D)\ |V\rangle\rangle=\hat{E}\ |V\rangle\rangle. \tag{17}$$

In the same reference it has been shown that Eq. (17) has a two-dimensional representation for $\alpha\neq\beta$ which can be simply shown that it is of form (15) with the parameters given in Eq. (7).

The reason that we emphasize the matrix representation of algebra (17) can be rewritten in the form of Eq. (15) (which is slightly different from what was first proposed in [10]) is as follows. As we have claimed in our previous papers, whenever the steady state of a one-dimensional driven-diffusive system defined on a finite or infinite open lattice which evolves under the random-sequential updating scheme can be written in terms of a linear superposition of Bernoulli shocks with simple random-walk dynamics, then the algebraic relation between the operators (when the steady state is studied using the matrix-product formalism) will have a two-dimensional representation with a generic structure. In [5] we have also proposed a general formalism by which one can simply find a two-dimensional representation for the quadratic algebra of the system in terms of the hopping rates of the shock front and the densities of the particles on the left- and the right-hand sides of the shock. This works if and only if the time evolution of the position of a product shock measure with a single shock front is simply a random walk. Moreover it has been shown, by providing several examples, that the conditions under which the domain wall has a random-walk dynamics are exactly those for the existence of the two-dimensional matrix representation [5].

VI. CONCLUDING REMARKS

Let us review the results of the current work. The most important goal in this work was answering the question of whether the matrix representation of the quadratic algebra of the system developing via sublattice-parallel updating scheme has the same generic structure as we had proposed for the case of continuous-time updating scheme. By comparing our results in this paper with those in [5] one finds that the matrix representation retains its structure even in sublattice-parallel updating scheme and it seems that, although we have no direct proof for it at the moment, the same is true for other updating schemes. In this direction we have considered a general driven-diffusive system with nearest-neighbor interactions which evolve via sublattice-parallel updating scheme and is defined on an open lattice. If we assume that a single product shock measure has a simple random-walk dynamics, generated by the transfer matrix of the system, then the steady state of this system can easily be written in terms of a linear superpositions these shock measures. On the other hand we have introduced a two-dimensional matrix representation which can generate exactly the same steady state. The nontrivial point is that this matrix representation has exactly the same structure that we had found for the case of continuous time updating scheme.

As an evidence we have studied the TASEP under the sublattice-parallel updating scheme and shown that an uncorrelated shock can evolve in the system without requiring any constraints on the microscopic reaction rates, i.e., the injection and the extraction rates of the particles. The shock also reflects from the boundaries of the lattice with some nonzero rates. By investigating the time evolution equations of the shock front, we have found that it has simple random-walk dynamics. Since the dynamics of the shock front is quite similar to that of a random walker [12], the steady state of the system can be constructed as a linear superposition of such product shock distributions. This could have been supposed since our experience with random-sequential updating scheme had shown that in this case the quadratic algebra of the system should have a two-dimensional matrix representation as it was found in [10]. As we have seen in this paper a two-dimensional matrix representation for the quadratic algebra of the TASEP under discrete time updating exists without any constraints.

Our investigations show that the ASEP with the most general four-parameter open boundary conditions studied in [13] can be explained using our approach provided that the same conditions under which the quadratic algebra of the system has a two-dimensional matrix representation are fulfilled. We have also found other families of driven-diffusive models evolving under sublattice-parallel updating scheme in which a product shock measure with a single shock front has a simple random-walk dynamics very similar to Eqs. (6), provided that some constraints on the microscopic reaction probabilities are satisfied. We have shown that the steady state of these systems can be written in terms of a combinations of such single shocks, and at the same time the matrix representation of the quadratic algebras of these systems has the same unique structure as in Eq. (15). The details of these results will be presented elsewhere.

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